Tips for faster calculation

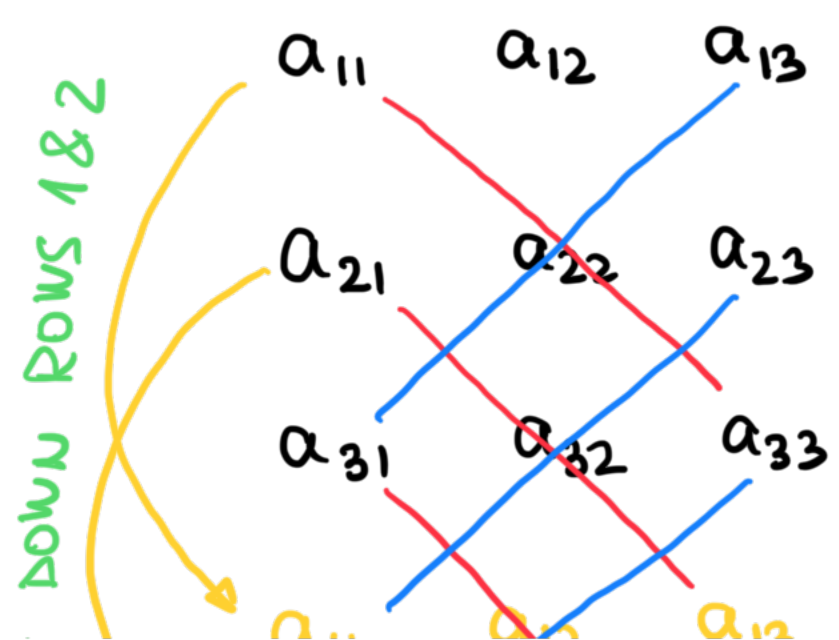
(always check your answer, to make sure ∄ no typos)

Determinants (formulas from class)

- 2×2: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = +ad - bc$

- 3×3: $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$

A mnemonic for remembering this formula is as follows



2: MULTIPLY ALONG DIAGONALS

$a_{11}a_{22}a_{33}$	$a_{13}a_{22}a_{31}$
$a_{21}a_{32}a_{13}$	$a_{23}a_{32}a_{11}$
$a_{31}a_{12}a_{23}$	$a_{33}a_{12}a_{21}$

3: ADD/SUBTRACT

1: COPY

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

$$+a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$
$$-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

Inverses of matrices:

- $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- $$\begin{pmatrix} a & x & z \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & u & w \\ 0 & \frac{1}{b} & v \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$$ (works for any triangular matrix)

  solve for u, v, w recursively (those closest to diagonal first) by multiplying the two matrices and setting equal to I_3

$$I_3 = \begin{pmatrix} a & x & z \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} \frac{1}{a} & u & w \\ 0 & \frac{1}{b} & v \\ 0 & 0 & \frac{1}{c} \end{pmatrix} \rightsquigarrow \begin{cases} 0 = au + \frac{x}{b} & \text{solve for } u \\ 0 = bv + \frac{y}{c} & \text{solve for } v \\ 0 = aw + xv + \frac{z}{c} & \text{solve for } w \end{cases}$$

- general A^{-1}: set up augmented matrix

$$(A \mid I_n) = \left(A \;\middle|\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right)$$ and do Gaussian elimination until left block is I_n

$$\Big\downarrow$$

$$\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \;\middle|\; A^{-1}\right)$$ what you find here is A^{-1}

(why does this work? Because Gaussian elimination corresponds

to left multiplication by some matrix X: $(A \mid I_n) \rightsquigarrow (XA \mid X) = (I_n \mid X)$
$$\Downarrow$$
$$X = A^{-1} \quad)$$

Note: the "cofactor formula" for inverse matrices is almost never efficient (except for really simple matrices with lots of 0's), which is why we don't teach it and recommend Gaussian elimination as above

$$\left(A^{-1}\right)_{ij} = \frac{(-1)^{i+j} \det(A \text{ with row } j \text{ and column } i \text{ removed})}{\det(A)}$$

Orthogonal bases

- in $\mathbb{R}^2$: a vector perpendicular to $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\begin{pmatrix} b \\ -a \end{pmatrix}$

- in $\mathbb{R}^3$ only! The cross product of vectors u and v gives you a vector that's orthogonal to both u and v; great for computing orthogonal bases

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \; v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow u \times v = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} \perp u, v$$

(ONLY WORKS IN $\mathbb{R}^3$)

Note: because the cross product was not part of this course, make sure you explain to the grader why it works on an open problem

"we learned that the cross product $u \times v$ is perpendicular to u, v; hence if $u \perp v$, then $u, v, u \times v$ form an orthogonal basis"

or better

"we can check by hand that $u \times v$ is perpendicular to u, v:

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \ u \times v = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \Rightarrow \begin{matrix} (u \times v) \cdot u = -3 + 12 - 9 = 0 \\ (u \times v) \cdot v = -12 + 30 - 18 = 0 \end{matrix}$$ "

Eyeballing (great for saving time on small examples), e.g.

find eigenvector for $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ and eigenvalue 3:

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} 1-3 & 1 \\ -2 & 4-3 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}$$; take $-2x + y = 0$, e.g. $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

check that this works: $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Note: though technically it should be avoided in an open problem, eyeballing might be accepted if you **prove** that the thing you eyeball works in whatever setting you're dealing with, e.g.

"let us prove that $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ corresponding to the eigenvalue $\lambda = 3$: $A v = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ hence $Av = \lambda v$"

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \begin{pmatrix} 3 \\ \ldots \end{pmatrix}$

$$\lambda v = 3\begin{pmatrix} \\ 2 \end{pmatrix} = \begin{pmatrix} \\ 6 \end{pmatrix}$$